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A CONGRUENCE OF CIRCLES.*

By Frederick Wahn Beal.

Introduction.

1. This paper treats of a particular type of two-parameter systems of circles whose planes envelope a non-developable surface S. The curvilinear coördinates on S will be u and v and the congruence of circles will be called a congruence C. Any point P of a circle of the congruence is to be a point P of a surface S_1, of a single-parameter family of surfaces. The tangent plane at P to S_1 is to pass through the center of the circle and to make an angle with the plane of the circle, which, at most, is a function of u and v. When this angle is a right angle, the system of circles is cyclic. The existence and some general properties of the congruence C will be proved first. Afterwards the particular case in which the center of the circle is the point at which its plane touches its envelope will be considered and some characteristic properties discovered. In this case the system of circles is called a congruence C_0. The system of circles defining the Backlund transformations of a pseudospherical surface is a congruence of this kind. Congruences C_0 characterize a class of surfaces to which belong also the surfaces S_1 as defined above. We say that each surface S_1 is a transform of S. The Bianchi and Backlund transformations† of pseudospherical surfaces are special cases of the more general transformations defined by congruences C_0.

I. General treatment of congruences C.

2. In order to attack the problem under consideration we will associate with S a trirectangular set of moving axes, in such a way that the z-axis is the normal to S and the xy-plane is the tangent plane to S. The coördinates $(a, b, 0)$ of the center of the circle relative to these axes, the radius R of the circle and the angle ϕ which the normal to S_1 at P makes with the plane of the circle will all, in general, be functions of u and v. The formulæ giving the total displacements of a point P are‡

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* Read before the American Mathematical Society, Apr. 24, 1915.

† Cf. Eisenhart, Differential Geometry, (Ginn & Co., Boston, 1909), p. 284. Further references to this work will be given in the form Eisenhart, p. 284.

‡ Eisenhart, p. 166.

$$(1)\quad \begin{cases} \delta x = dx + \xi du + \xi_1 dv + (q du + q_1 dv)z - (r du + r_1 dv)y, \\ \delta y = dy + \eta du + \eta_1 dv - (p du + p_1 dv)z + (r du + r_1 dv)x, \\ \delta z = dz + (p du + p_1 dv)y - (q du + q_1 dv)x, \end{cases}$$

where δx and dx denote total and relative displacements respectively, and ξ, ξ_1, η, η_1, are the *translations* and p, p_1, q, q_1, r, r_1 the *rotations* of the tri-rectangular axes.

The coördinates of a point P on the circle are

$$(a + R \cos \theta,\ b + R \sin \theta,\ 0),$$

and the direction-cosines of the normal to S_1 at P are

$$\cos \phi \sin \theta, \qquad -\cos \phi \cos \theta, \qquad \sin \phi.$$

If the motion of P is to be perpendicular to this direction for all displacements on S in the neighborhood of M, the vertex of the trihedral, the following equation must be satisfied:

$$\cos \phi \sin \theta\, \delta x - \cos \phi \cos \theta\, \delta y + \sin \phi\, \delta z = 0.$$

This equation reduces to

$$(2)\quad \begin{aligned} &R \cos \phi d\theta + [\cos \phi(B \cos \theta - A \sin \theta + Rr) + \sin \phi(qa + qR \cos \theta \\ &\quad - pb - Rp \sin \theta)]du + [\cos \phi(B_1 \cos \theta - A_1 \sin \theta + Rr_1) \\ &\quad + \sin \phi(q_1 a + q_1 R \cos \theta - p_1 b - p_1 R \sin \theta)]dv = 0, \end{aligned}$$

where A_i and B_i are defined by the following:

$$A_1 = a_v + \xi_1 - r_1 b, \qquad B_1 = b_v + \eta_1 + r_1 a,$$
$$A = a_u + \xi - rb, \qquad B = b_u + \eta + ra.\text{*}$$

By the substitution $t = \tan \theta/2$, this equation becomes a Riccati equation and we have, as is true for cyclic systems, that

THEOREM. *Any four surfaces of the system S_1 meet the circles in four points whose cross-ratio is constant.*

If equation (2) is to possess a solution $\theta(u, v)$, it must satisfy a well-known condition of integrability,† which in this case reduces to the form

$$(3)\quad T \cos \theta + U \sin \theta + V = 0,$$

where T, U and V are functions of u and v. This equation, if it is not an

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[*] Throughout the paper derivatives with respect to u and v are indicated by subscripts, as a_u and a_v.

[†] Forsyth, a Treatise on Differential Equations (Macmillan & Co., 1903), p. 282.

identity, possesses at most two distinct solutions which may be real and satisfy equation (2). Hence again as for cyclic systems we have that

THEOREM. *If more than two surfaces S_1 exist, there exist a single infinity and the congruence of circles is a congruence C.*

The necessary and sufficient condition that the system of circles be a congruence C is that θ involve an arbitrary parameter. This means that equation (3) is an identity in θ, and gives the following three equations of condition:

$$\begin{aligned} Rb(pq_1 - p_1q) + (AR_v - A_1R_u)\cos^2\phi - a\sin\phi\cos\phi(qB_1 - q_1B) \\ + b\sin\phi\cos\phi(pB_1 - p_1B) - R^2(p\phi_v - p_1\phi_u) = 0, \end{aligned} \tag{4}$$

$$\begin{aligned} Ra(pq_1 - p_1q) + (B_1R_u - BR_v)\cos^2\phi + a\sin\phi\cos\phi(q_1A - qA_1) \\ + b\sin\phi\cos\phi(pA_1 - p_1A) + R^2(q\phi_v - q_1\phi_u) = 0, \end{aligned} \tag{5}$$

$$\begin{aligned} R^2(pq_1 - p_1q) + (AB_1 - A_1B)\cos^2\phi + a\sin\phi\cos\phi(q_1R_u - qR_v) \\ + b\sin\phi\cos\phi(pR_v - p_1R_u) + Ra(q\phi_v - q_1\phi_u) \\ - Rb(p\phi_v - p_1\phi_u) = 0. \end{aligned} \tag{6}$$

These equations may be readily solved for first order partial derivatives, with respect to u, of three of the four quantities a, b, R, and ϕ, unless ϕ is a right angle, in which case the system is cyclic. Hence, in general, one of the four functions is arbitrary and the other three are given by a system of first order partial differential equations which may be put in the type form. A relation may be assumed between a and b which will cause the center of the circle to generate a particular surface.

3. We will close the discussion of the general congruence C by making a few statements that can be easily verified. They will serve to emphasize the differences between the general case and the particular case of cyclic systems.

None of the following three conditions, all of which are satisfied by cyclic systems, is satisfied by a general system C, but without imposing any restrictions on S, the functions a, b, R, and ϕ can be determined so that anyone of these conditions can be satisfied by a congruence C:

The developables of the congruence of axes of the circles correspond to a conjugate system on S.

The lines joining a point of a circle to the focal points of the congruence of axes of the circles intersect under a right angle. For cyclic systems these lines are the tangents considered in the next statement.

The tangents to S_1 at P that pass through the projections on the tangent plane to S_1 at P of the focal points of the associated congruence of axis intersect at right angles. This condition is satisfied by congruences C_0.

II. Congruences C_0.

When a and b are identically zero, we will call the envelope of the planes of the circles the surface A. The corresponding congruence of circles has been called a congruence C_0. If the lines of curvature on A are taken as parametric and the x-axis of the trihedral tangent to $v = \text{const.}$, the conditions (4), (5) and (6) reduce to the following:

$$qR_v + \eta_1\phi_u = 0, \tag{7}$$

$$p_1R_u + \xi\phi_v = 0, \tag{8}$$

$$R^2K + \cos^2\phi = 0, \tag{9}$$

where K denotes the total curvature of the surface A.* When the equations obtained from (7) and (8) by eliminating R are solved for the derivatives of ϕ, we have the following:

$$\phi_v = \frac{1}{2}\left[\sec\phi\sqrt{\frac{p_1\eta_1}{q\xi}}\,\frac{\partial}{\partial u}\log K + \tan\phi\,\frac{\partial}{\partial v}\log K\right], \tag{10}$$

$$\phi_u = \frac{1}{2}\left[\sec\phi\sqrt{\frac{q\xi}{p_1\eta_1}}\,\frac{\partial}{\partial v}\log K + \tan\phi\,\frac{\partial}{\partial u}\log K\right], \tag{11}$$

The condition of integrability of these equations is reducible to

$$\begin{aligned}&\frac{\partial}{\partial u}\,\frac{p_1\eta_1}{q\xi}\,\frac{\partial}{\partial u}\log K + 2\frac{p_1\eta_1}{q\xi}\,\frac{\partial^2}{\partial u}\log K - \frac{p_1\eta_1}{q\xi}\left[\frac{\partial}{\partial u}\log K\right]^2\\ &\qquad = \frac{\partial}{\partial v}\log\frac{q\xi}{p_1\eta_1}\,\frac{\partial}{\partial v}\log K + 2\frac{\partial^2}{\partial v^2}\log K - \left[\frac{\partial}{\partial v}\log K\right]^2.\end{aligned} \tag{12}$$

This equation characterizes the A surfaces.

That pseudospherical surfaces are A surfaces is evident. To show that other A surfaces exist, we have obtained the following fundamental quantities of a surface of revolution of total curvature $-1/au$ which satisfy equation (12) and the six equations,† equivalent to the Gauss and Codazzi equations of condition:

$$\xi_1 = \eta = r = p = q_1 = 0, \qquad \xi = p_1 = U_1, \qquad \eta_1 = U_2,$$

$$r_1 = \frac{d}{du}\log U_2, \qquad q = \frac{U_2}{au}.$$

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* Eisenhart, p. 172.

† Eisenhart, pp. 168, 170.

The quantities U_1 and U_2 are functions of u alone and satisfy the following ordinary differential equations:

$$auU_1U_1' + U_2U_2' = 0,$$

$$auU_1U_2'' - auU_2'U_1' - U_1^3U_2 = 0.$$

5. The coördinates of a point P on the normal to a surface S_1, which we will now call A_1, are

$$(R \cos \theta + t \cos \phi \sin \theta, \qquad R \sin \theta - t \cos \phi \cos \theta, \qquad t \sin \phi),$$

where t is the distance from the surface A_1 to P. The conditions that P be a focal point of this congruence of normals, *i. e.*, that P move tangentially to the normal, are

$$\frac{\delta x}{\cos \phi \sin \theta} = \frac{\delta y}{-\cos \phi \cos \theta} = \frac{\delta z}{\sin \phi},$$

where δx, δy, and δz are the total displacements of P given by (1). These conditions reduce to the equations

$$\begin{aligned}&[RR_u + t\xi \sin \theta \cos \phi + \xi R \cos \theta]du + [RR_v - t\eta_1 \cos \theta \cos \phi \\ &\qquad\qquad + \eta_1 R \sin \theta]dv = 0,\\ &[RR_u \sin \phi \cos \phi \cos \theta - R\xi \sin \phi \cos \phi \sin^2 \theta \\ &\quad + t\xi \sin \phi \cos^2 \phi \sin \theta \cos \theta + \xi R \sin \phi \cos \theta - tR\phi_u \cos \phi \sin \theta \\ &\quad + qR^2 \sin \theta \cos \theta + qtR \cos \phi \sin^2 \theta]du + [RR_v \sin \phi \cos \phi \cos \theta \\ &\quad + \eta_1 R \sin \phi \cos \phi \sin \theta \cos \theta - t\eta_1 \sin \phi \cos^2 \phi \cos^2 \theta - tR\phi_v \cos \phi \sin \theta \\ &\quad - p_1R^2 \sin^2 \theta + p_1tR \cos \phi \sin \theta \cos \theta]dv = 0.\end{aligned}$$

If du and dv are eliminated between these equations, a quadratic in t is obtained for which the coefficient of t^2 and the term not involving t are respectively

$$R \cos^2 \phi \sin \theta[p_1 \sin \theta R_u + q \cos \theta R_v + \sin \theta \cos \theta(p_1\xi + q\eta_1)]$$

and

$$-R^3 \sin \theta[p_1 \sin \theta R_u + q \cos \theta R_v + \sin \theta \cos \theta(p_1\xi + q\eta_1)].$$

Hence the total curvature of A_1, that is the reciprocal of the product of the roots, is $-(\cos^2 \phi)/R^2$. Since this expression does not involve θ it is the same for all the surfaces A_1. Moreover, from (9) it follows that it is also the total curvature of A. Hence we have

THEOREM. *The congruence C_0 defines a transformation of a surface A into a single infinity of surfaces A_1 of the same total curvature as A.*

The surfaces A and A_1 are evidently the focal surfaces of the congruence of lines joining A and A_1. The necessary and sufficient condition that this congruence be normal is that the tangent planes to A and A_1 be perpendicular, *i. e.*, $\phi = 0$.

Under this condition equations (7), (8) and (9) show that A and hence A_1 are pseudospherical surfaces. The surfaces A_1 are Bianchi transforms of the pseudospherical surface A in this case.

6. The linear element of the surfaces A_1 is

$$\begin{aligned} dS^2 = (\xi^2 \cos^2\theta + R_u^2 + 2\xi\cos\theta R_u + q^2R^2\cos^2\theta\sec^2\phi)du^2 \\ + 2(R_uR_v + \xi\cos\theta R_v + \eta_1\sin\theta R_u)dudv \\ + (\eta_1^2\sin^2\theta + R_v^2 + 2\eta_1\sin\theta R_v + p_1^2R^2\sin^2\theta\sec^2\phi)dv^2. \end{aligned}$$

Not more than two of these surfaces can have an orthogonal parametric system in correspondence with the lines of curvature on surface A unless R is constant. When R is constant, ϕ and K are both constant, and the surfaces A_1 are Backlund transforms of the pseudospherical surface A. Under these conditions the parametric curves on A_1 are lines of curvature. These conditions together with equations (7), (8), (9), (10), and (11) give the following:

THEOREM. *If one of the following four conditions is true, the other three are also true*; ϕ, R, *or* K *constant, lines of curvature on surface* A *correspond to lines of curvature on surfaces* A_1.

III. **Transformations of surfaces A.**

7. The Bianchi and Backlund transformations of a pseudospherical surface yield pseudospherical surfaces, which are reciprocally related to the original surface in the sense that the latter is one of the Bianchi or Backlund transforms of the transformed surfaces. We will now show that the surfaces A_1 obtained from A under the more general transformation C_0 are so related to A.

We assume that the surface A is subjected to one of these transformations and that the resulting surface is A_1. The lines joining corresponding points on A and A_1 are tangent to a one parameter family of curves on A, which we take for the curves $v =$ const. We choose for the curves $u =$ const. the conjugate system to $v =$ const. If the moving trihedral is chosen so that the x-axis is tangent to the curves $v =$ const. we have*

$$\eta = q_1 = 0, \qquad p\eta_1 = q\xi_1.$$

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* Eisenhart, p. 173.

Furthermore, from the choice of $v = \text{const.}$, it follows that the function θ for the transformation is equal to zero. Hence from equation (2) we have

$$q \tan \phi + r = 0, \qquad \eta_1 + Rr_1 = 0. \tag{13}$$

Equations (4), (5), and (6) then reduce to the two equations

$$\eta_1 p_1(r^2 + q^2) = \xi q r_1^2, \tag{14}$$

$$q^2 \xi(\eta_1 r_{1v} - r_1 \eta_{1v}) + \eta_1^2 p_1(r q_u - q r_u) = 0. \tag{15}$$

To compute the fundamental quantities of A_1, we associate with A_1 a moving trihedral whose x-axis is the x-axis of the trihedral for A and has the same direction. The vertices of these trihedrals are M and P. Since A and A_1 are focal surfaces of the congruence of lines made up of the x-axes and since a conjugate system of curves is parametric on A, a conjugate system of curves is also parametric on A_1. If the fundamental quantities of A_1 are indicated by primes, both sides of the first three of the following equations give the total displacements of P in the direction of the axes of the trihedral at P. The last three equations give the total displacements of M in the direction of the axes at M.

$$\xi' du + \xi_1' dv = (R_u + \xi) du + (R_v + \xi_1) dv,$$

$$\eta' du + \eta_1' dv = (Rr \sin \phi - qR \cos \phi) du,$$

$$(Rr \cos \phi + qR \sin \phi) du - \cos \phi (\eta_1 + r_1 R) dv = 0,$$

$$\xi du + \xi_1 dv = (-R_u + \xi') du + (-R_v + \xi_1') dv,$$

$$\eta_1 dv = [(\eta' - Rr') \sin \phi - q'R \cos \phi] du + [-Rr_1' \sin \phi - \cos \phi q_1' R] dv,$$

$$[(\eta' - Rr') \cos \phi + q'R \sin \phi] du + [-Rr_1' \cos \phi + q_1' R \sin \phi] dv = 0.$$

These in virtue of (13) and the equality of the total curvatures of A and A_1 at M and P yield the following:

$$\begin{array}{lll} \xi' = R_u + \xi, & \eta' = R(r \sin \phi - q \cos \phi), & \eta_1' = q' = 0, \\ \xi_1' = R_v + \xi_1, & r' = r \sin \phi - q \cos \phi, & r_1' = r_1 \sin \phi, \\ p' = \dfrac{q}{\eta_1}(R_v + \xi_1), & p_1' = \dfrac{p_1}{\xi}(R_u + \xi), & q_1' = r_1 \cos \phi. \end{array}$$

In order that a system of circles of radius R, with center at P and lying in the tangent planes of A_1 possess a single infinity of transformed surfaces, one of which is A, whose normals make the angle ϕ with the tangent planes to A_1 and are perpendicular to the radius of the circle, it is necessary and sufficient that equations analogous to (4), (5), and (6) be satisfied and that $\theta = \pi$ be a solution of an equation analogous to (2). These conditions, with the exception of the equation analogous to (4), may easily be shown to be satisfied. Equation (4) reduces to

$$(\xi R_v - \xi_1 R_u) \cos^2 \phi - R^2(p'\phi_v - p_1'\phi_u) = 0.$$

A similar condition for the surface A is

$$(\xi R_v - \xi_1 R_u) \cos^2 \phi - R^2(p\phi_v - p_1\phi_u) = 0.$$

From these we have that the desired condition is

$$(p - p')\phi_v + (p_1' - p_1)\phi_u = 0.$$

This may be reduced to either one of the two equivalent equations

$$\phi_v R_v = \phi_u R_u \sin^2 \phi,$$

$$\eta_1 p_1(rq_u - qr_u)[p_1\eta_1(r^2 + q^2) - q\xi r_1^2] = 0.$$

The first of these gives a relation between R and ϕ and their derivatives. The second which is equivalent to the first is satisfied by virtue of equation (14).

From the preceding we are able to state the following results:*

THEOREM. *Every surface A_1, obtained from the surface A by means of the congruence C_0, admits of a like congruence C_0. In each case one of the transforms of A_1 is A.*

THEOREM. *Every congruence C_0 has associated with it a complex of circles which is composed of congruences C_0.*

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* The latter part of abstract 19 on page 490 of the Bulletin of the American Mathematical Society for July, 1915, is incorrect.